FINANCIAL MATHEMATICS.

You have already used the following formulae in Grade 11 :

Simple interest formula: $A = P(1 \pm in)$

Compound interest formula : $A = P(1 \pm i)^n$

A = the future value of the money invested
P = the initial amount invested
i = interest rate as a decimal fraction
n = number of interest bearing periods

It is a good idea first to revise the section of Financial Mathematics in the Grade 11 study guide.

1. Calculation of the investment period (n) in the formula $A = P(1 \pm i)^n$:

In Grade 11, you were unable to calculate the period (n) when working with compound interest. However, now that you have been introduced to logarithms you will be able to calculate the value of n. Consider the following example.

Example 1.

Lesiba was awarded a bonus of R4 500 for excellence at his workplace. He invested this money at a rate of 10% per annum compounded annually and received R9 646,12 at the end of the period. For how many years was the money invested ?

Solution :

Compound interest $\therefore\ A = P(1+i)^n$

$\therefore\ 9646,12 = 4500(1+0,1)^n \qquad (A = 9646,12 : P = 4500 : i = 0,1)$

$\therefore\ \dfrac{9646,12}{4500} = (1+0,1)^n \qquad (\textit{Divide by } 4500.)$

$\therefore\ 2,14358.... = (1,1)^n \qquad (\textit{Simplify LHS and inside brackets on RHS.})$

$\therefore\ \log 2,14358. = \log(1,1)^n \qquad (\textit{Take } \text{logs } \textit{on both sides.})$

$\therefore\ \log 2,14358.. = n \log 1,1 \qquad (\log a^n = n \log a.)$

$$\therefore \; n = \frac{\log 2,14358..}{\log 1,1}$$

$\therefore \; n = 7,9999$ $\qquad$ <u>$\therefore$ *The money was invested for* 8 *years*.</u>

<u>Example 2.</u>

A man invests R15 000 at 12% per annum, compounded monthly. At the end of the period he receives R27 250. For what period did he invest the money ?

<u>Solution :</u>

$$A = (1+i)^n$$

$$\therefore \; 27250 = 15000(1 + \frac{0,12}{12})^{12n}$$ (*Compounded monthly* $\therefore$ *divide interest rate by* 12 *and multiply n by* 12.)

$$\therefore \; \frac{27250}{15000} = \left(1 + \frac{0,12}{12}\right)^{12n}$$ (*Divide by* 15000.)

$$\therefore \; 1,81666... = (1,01)^{12n}$$

$$\therefore \; \log 1,81666.. = \log(1,01)^{12n}$$ (*Take* logs *on both sides*.)

$$\therefore \; \log 1,81666.. = 12n \log(1,01)$$

$$\therefore \; 12n = \frac{\log 1,816666}{\log 1,01}$$

$$\therefore \; 12n = 59,9979$$

$$\therefore \; n = 4,9998$$ (*Divide by* 12.)

<u>$\therefore$ *Money was invested for* 5 *years.*</u>

<u>Example 3.</u>

Office equipment depreciates at 16% per annum on the reducing balance method. How long will it take before the value of the office equipment is halved ?

<u>Solution :</u>

Reducing balance depreciation $\therefore \; A = P(1-i)^n$

Value halved $\therefore$ *If* $P = 2x$, *then* $A = x$.

$$\therefore\ x = 2x(1 - 0,16)^n$$

$$\therefore\ \frac{x}{2x} = (1 - 0,16)^n \qquad (\textit{Divide by } 2x.)$$

$$\therefore\ 0.5 = (0,84)^n$$

$$\therefore\ \log(0,5) = \log(0,84)^n \qquad (\textit{Take } \log\text{s} \textit{ on both sides.})$$

$$\therefore\ \log(0,5) = n\log(0,84)$$

$$\therefore\ n = \frac{\log(0,5)}{\log(0,84)}$$

$$\therefore\ n = 3,9755$$

$\therefore$ *The value will be halved after* 4 *years.*

2. Future value of an annuity :

When people save money for a specific reason, they do not always invest a single initial amount, but deposit fixed monthly amounts into a savings account or an investment fund. Compound interest is paid on the money deposited into the fund. **The future value of an annuity** is the sum of all the deposits made, plus all the interest earned.

Consider the following examples.

Example 4.

Peter would like to buy a new television set in six months' time. He opens a savings account, and immediately deposits R500 into the account. He continues making monthly deposits of R500, and makes the last deposit one month before the end of the six-month period. The interest rate on his savings account is 12% per annum, compounded monthly.

What will the total balance of his savings account be after six months ?

Solution :

The interest rate is 12%, *compounded monthly* $\therefore \ \frac{i}{12} = \frac{0,12}{12} = 0,01$

On the first deposit he will recieve interest for 6 *months.*

$\therefore \ 500(1+0,01)^6 = 500(1,01)^6$

On the second deposit : $500(1,01)^5$ *etc.*

Let's place the deposits Peter made on a number line.

0	1	2	3	4	5	6 months
$500(1,01)^6$	$500(1,01)^5$	$500(1,01)^4$	$500(1,01)^3$	$500(1,01)^2$	$500(1,01)$	

Therefore, the total amount available after 6 *months will be* :

$$500(1,01) + 500(1,01)^2 + 500(1,01)^3 + 500(1,01)^4 + 500(1,01)^5 + 500(1,01)^6$$

This is a geometric series with : $a = 500(1,01)$

$$\text{and} \qquad r = (1,01) \qquad [r = \frac{T_2}{T_1} = \frac{500(1,01)^2}{500(1,01)} = 1,01]$$

Now calculate the sum of the first 6 *terms.*

$$S_n = \frac{a(r^n - 1)}{r - 1} \qquad \therefore \ S_6 = \frac{500(1,01)[(1,01)^6 - 1]}{1,01 - 1} \qquad = \underline{R3\ 106,77}$$

Example 5.

You start working and decide to deposit R1 000 into a savings account each year. The interest rate is 12% per annum, compounded annually. What will the account balance be after 4 years ? The first deposit was made at the end of the first year, and the last deposit at the end of the fourth year.

Solution :

Let's use the number line again.

0 1 2 3 4 Years

R1000 R1000 R1000 R1000

The first amount was only invested at the end of the first year, therefore for 3 *years.*

The first deposit earns interest for 3 *years* : $\therefore\ A = 1000(1 + 0,12)^3$
The second deposit earns interest for 2 *years* : $\therefore\ A = 1000(1 + 0,12)^2$
The third deposit earns interest for 1 *year* : $\therefore\ A = 1000(1 + 0,12)$

The last deposit was only made at the end of the investment period.
The last deposit therefore earns no interest : $\therefore\ A = 1000$

Now add all the amounts :

$1000 + 1000(1,12) + 1000(1,12)^2 + 1000(1,12)^3$ $[(1 + 0,12) - 1,12]$

This is a geometric series with :

To calculate the total amount after 4 *years, we can use the formula for the sum of a geometric series* : $S_n = \dfrac{a(r^n - 1)}{r - 1}$

$$\therefore\ S_4 = \frac{1000(1,12^4 - 1)}{1,12 - 1} \qquad (n = 4)$$

$$\therefore\ S_4 = R4779,33$$

Note the difference between the two examples. In example 4 no payment was made at the end of the period, whereas in example 5 there was a payment made at the end of the period.

The following formula is useful when we want to calculate the future value of an annuity or a savings plan :

<u>The future value annuity formula</u> :

$$F = \frac{x[(1 + i)^n - 1]}{i}$$ where $F = Future\ value$
$x = fixed\ payments\ per\ period.$
$i = interest\ rate, as\ a\ decimal\ fraction.$
$n = number\ of\ payments.$

However, this formula can only be used if a **final payment**, **that does not earn any interest**, **is made at the end of the period**, as in example 5.