QUADRATIC EQUATIONS AND INEQUALITIES.

1. **Solving quadratic equations** :

Quadratic equations can be solved in 4 different ways :

1.1 By factorising : (Already done in grade 10.)

Example 1.

Solve for x :

$$3(x^2+1) = 2(x+2)$$

$\therefore\ 3x^2+3 = 2x+4$ (*Remove brackets.*)

$\therefore\ 3x^2-2x+3-4 = 0$ (*Move all the terms to the left-hand side.*)

$\therefore\ 3x^2-2x-1 = 0$ (*Simplify left-hand side.*)

$\therefore\ (3x+1)(x-1) = 0$ (*Factorise.*)

$\therefore\ 3x+1 = 0$ *or* $x-1 = 0$

$\therefore\ 3x = -1$ *or* $\underline{x = 1}$

$\therefore \underline{x = -\frac{1}{3}}$

Example 2 .

Solve for x :

$$\frac{2x-1}{x-3} - \frac{8x-6}{x^2-9} = \frac{3x-4}{x+3}$$ (*Factorise denominators to find LCM.*)

$\therefore\ \frac{2x-1}{x-3} - \frac{8x-6}{(x-3)(x+3)} = \frac{3x-4}{x+3}$ [$LCM = (x-3)(x+3)$.]

$\therefore\ (2x-1)(x+3)-(8x-6) = (3x-4)(x-3)$ (*Multiply by LCM.*)

$\therefore\ 2x^2+6x-x-3-8x+6 = 3x^2-9x-4x+12$ (*Remove brackets.*)

$\therefore\ 2x^2 - 3x + 3 = 3x^2 - 13x + 12$ (*Simplify left-hand and right-hand side.*)

$\therefore\ 2x^2 - 3x^2 - 3x + 13x + 3 - 12 = 0$ (*Move all the terms to the left-hand side.*)

$\therefore\ -x^2 + 10x - 9 = 0$ (*Add like terms.*)

$\therefore\ x^2 - 10x + 9 = 0$ (*Divide by* -1.)

$\therefore\ (x - 9)(x - 1) = 0$ (*Factorise left-hand side.*)

$\underline{\therefore\ x = 9}$ *or* $\underline{x = 1}$

1.2. Using the k method : (Also done in grade 10.)

Example 3.

Solve for a :

$$\frac{a^2 - a - 3}{a^2 - a} + a^2 - a + 1 = \frac{5}{a^2 - a}$$

Note : When you take $a^2 - a$ as LCM and then multiply by $a^2 - a$ you will get a^4 and a^3. We only solve quadratic equations. However, you can substitute $a^2 - a$ for k, e.g., and then the expression becomes :

$$\frac{k - 3}{k} + k + 1 = \frac{5}{k}$$ (*Now the LCM is* $\rightarrow k$.)

$\therefore\ k - 3 + k(k + 1) = 5$ (*Multiply by* k.)

$\therefore\ k - 3 + k^2 + k - 5 = 5$

$\therefore\ k^2 + 2k - 8 = 0$

$\therefore\ (k - 2)(k + 4) = 0$ (*Then substitute* k *for* $a^2 - a$.)

$\therefore\ (a^2 - a - 2)(a^2 - a + 4) = 0$

$\therefore\ (a^2 - a - 2) = 0$ *or* $(a^2 - a + 4) = 0$

$\therefore\ (a - 2)(a + 1) = 0$ *or* $a^2 - a + 4$ *has no factors – in example* 5 *the use of the formula is explained.*

$\therefore\ \underline{a = 2}$ *or* $a = -1$

1.3. Completing the square :

Example 4 A.

Solve for y by completing the square :

$2y^2 - 3y - 7 = 0$

$\therefore\ y^2 - \frac{3}{2}y - \frac{7}{2} = 0$ (*Divide by the coefficient of* y^2; *in this case* 2.)

$\therefore\ y^2 - \frac{3}{2}y = \frac{7}{2}$ (*Move constant term to right-hand side.*)

$\therefore\ y^2 - \frac{3}{2}y + (-\frac{3}{4})^2 = \frac{7}{2} + (-\frac{3}{4})^2$ [*Add to each side* : $(\frac{1}{2}$ *of coefficient of* $y)^2$, *therefore* $\rightarrow \{\frac{1}{2}(-\frac{3}{2})\}^2\ \therefore\ (-\frac{3}{4})^2$.]

$\therefore\ (y - \frac{3}{4})^2 = \frac{7}{2} + \frac{9}{16}$ (*Factorise left-hand side and square* $(-\frac{3}{4})$ *on right-hand side.*)

$\therefore\ (y - \frac{3}{4})^2 = \frac{56+9}{16}$ (*Find LCM on right-hand side* $\longrightarrow$ *you may use your calculator.*)

$\therefore\ (y - \frac{3}{4})^2 = \frac{65}{16}$

$\therefore\ y - \frac{3}{4} = \frac{\pm\sqrt{65}}{4}$ (*Find square roots on both left-hand and right-hand side.*)

$\therefore\ y = \dfrac{3 \pm \sqrt{65}}{4}$ (*Move* $-\frac{3}{4}$ *to right-hand side and write down on one denominator.*)

$\therefore\ \underline{y = 2,77}$ *or* $\underline{y = -1,27}$ (*Use calculator.*)

Example 4B.

Solve for x by completing the square :

$ax^2 + bx + c = 0$

$\therefore\ x^2 + \dfrac{b}{a}x + \dfrac{c}{a} = 0$ (*Divide by* a, *the coefficient of* x^2.)

$\therefore\ x^2 + \dfrac{b}{a}x = -\dfrac{c}{a}$ (*Move constant term to right-hand side.*)

$\therefore\ x^2 + \dfrac{b}{a}x + (\dfrac{b}{2a})^2 = (\dfrac{b}{2a})^2 - \dfrac{c}{a}$ (*Add* $[\frac{1}{2}(\frac{b}{a})]^2$ *to both left-hand and right-hand side.*)

$\therefore\ (x + \dfrac{b}{2a})^2 = \dfrac{b^2}{4a^2} - \dfrac{c}{a}$ (*Factorise left-hand side and square* $(\dfrac{b}{2a})$ *on right-hand side.*)

$\therefore\ (x + \dfrac{b}{2a})^2 = \dfrac{b^2 - 4ac}{4a^2}$ (*Find LCM on right-hand side and simplify.*)

$\therefore\ x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$ (*Find square root on both left-hand and right-hand side.*)

$\therefore\ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ ($\frac{b}{2a}$ *to right-hand side and write down on one denominator.*)

This now brings us to the formula used to solve any quadratic equation that has no factors. A quadratic equation is always in the form of $ax^2 + bx + c = 0$ so the roots of the equation are :

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

1.4. Using the formula :

Example 5.

Solve for x, correct to two decimal places :

$3x^2 - 2x - 4 = 0$ (*No factors; so the formula must be used.*)

$\therefore\ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ ($a = 3; b = -2; c = -4.$)

$\therefore\ x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-4)}}{2(3)}$

$\therefore\ x = \frac{2 \pm \sqrt{52}}{6}$ (*Use your calculator.*)

$\therefore\ \underline{x = 1,54}$ *or* $\underline{x = -0,87}$

Example 6.

6.1 Solve for a :

$a + \frac{1}{a} = 2$

$\therefore\ a^2 + 1 = 2a$ (*Multiply by LCM → a.*)

$\therefore\ a^2 - 2a + 1 = 0$ (*All the terms to left-hand side and write as trinomial.*)

$\therefore\ (a - 1)(a - 1)$ (*Factorise.*)

$\therefore\ a - 1 = 0$ *or* $a - 1 = 0$ $\underline{\therefore\ a = 1}$

6.2 Hence, solve for x (round off answer to 2 decimal places, if necessary) :

$(2x^2 + 3x) + \dfrac{1}{(2x^2 + 3x)} = 2$ [*Note : in* 6.2 *a has been replaced by* $(2x^2 + 3x)$, *so replace a by* $(x^2 + 3x)$]

$\therefore\ 2x^2 + 3x = 1$ [*In* 6.1 $a = 1$ $\therefore$ $(2x^2 + 3x) = 1 \rightarrow$ *the same as k method*.]

$\therefore\ 2x^2 + 3x - 1 = 0$ (*No factors* → *Use formula*.)

$$\therefore\ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore\ x = \frac{-3 \pm \sqrt{(3)^2 - 4(2)(-1)}}{2(2)}$$

$$\therefore\ x = \frac{-3 \pm \sqrt{17}}{4}$$

$\therefore\ x = 0.28$ *or* $x = -1.78$

2. Solving quadratic inequalities :

When solving quadratic equations, we multiply by the LCM, and then the LCM may be left out. However, when we work with a quadratic inequality, **the LCM must be retained** because, even though the LCM may not become zero, it will still affect the sign of the function.

Example 7.

Solve for x :

$x(x - 3) \geqslant 4$

$\therefore\ x^2 - 3x - 4 \geqslant 0$ (*Remove brackets and move* 4 *to the left-hand side* → *Right-hand side must be* 0.)

$\therefore\ (x - 4)(x + 1) \geqslant 0$ (*Factorise*.) *Zeros are* $x = 4$ *or* $x = -1$.

Method 1 :

Show zeros on number line, introduce a number to the left of -1 *to determine the sign of the function. Change the sign of the function at each zero.*

Let $x = -2$ $\therefore$ $(-2 - 4)(-2 + 1) = (-6)(-1) = 6$ *which is positive.*